Graphs and Combinatorics (2023) 39:16
https://doi.org/10.1007/s00373-023-02613-3

ORIGINAL PAPER

A Comparison of the Grundy and b-Chromatic Number of $K_{2,t}$ -Free Graphs

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Received: 10 December 2021 / Revised: 22 December 2022 / Accepted: 3 January 2023 /
Published online: 27 January 2023

Abstract

The Grundy and b-chromatic numbers of graphs are two important chromatic parameters. The Grundy number of a graph G, denoted by $\Gamma(G)$ is the worst case behavior of the greedy (First-Fit) coloring procedure for G and the b-chromatic number $\mathrm{b}(G)$ is the maximum number of colors used in any color-dominating coloring of G. There is a recent research scenario to compare these completely different chromatic quantities. A graph G is called b-monotone if $\mathrm{b}(H) \leq \mathrm{b}(G)$, for any induced subgraph H of G. A previous conjecture claims that there exits a function f such that for any b-monotone graph G, $\Gamma(G) \leq f(\mathrm{b}(G))$. We first prove that the family of $K_{2,t}$-free graphs is not b-monotone, and there does not exist a function f such that $\Gamma(G) \leq f(\mathrm{b}(G))$ for all $K_{2,t}$-free graphs G. We show that the collection of $K_{2,t}$-free graphs includes many infinite b-monotone subfamilies. Then, for any $t \geq 2$ we present a function f (dependent on t) such that if G is any b-monotone $K_{2,t}$-free graph then $\Gamma(G) \leq f(\mathrm{b}(G))$. Next, we obtain much better functions f for the families of $K_{2,2}$-free, $(K_{2,t}, K_3)$-free and $(K_{2,t}, K_4)$-free graphs. Putting differently, the results present inequalities in terms of $\Gamma(G)$, $\mathrm{b}(G)$ and t, where t is the smallest value such that G does not contain $K_{2,t}$ as induced subgraph.

Keywords Graph coloring · First-Fit coloring · Grundy number · Color-dominating coloring · b-chromatic number · b-monotone · $K_{2,t}$-free graphs

Mathematics Subject Classification 05C15 · 05C35

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1 Introduction

All graphs in this paper are undirected without any loops or multiple edges and unless otherwise stated, contain at least one edge. We refer to [1] for notations and concepts not defined here. Let $H_1, \ldots, H_k$ be a finite set of graphs. By a $(H_1, \ldots, H_k)$-free graph we mean any graph G such that for each i, G does not contain H_i as induced subgraph. Also H-free graph means any graph containing no induced subgraph isomorphic to H. For any graph G and vertex v of G, the closed neighborhood of v, $N[v]$, is the set $N(v)$ of neighbors of v together with v itself. For any $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S. For any positive integers m, n, $K_{m,n}$ stands for the complete bipartite graph with bipartition of sizes n and m. Given two positive integers r and s, recall that the Ramsey number $R(r, s)$ is the minimum value n such that any graph on n vertices either contains a clique on r vertices or an independent set containing s vertices. By $\log k$, we mean $\log_2 k$.

By a Grundy coloring of a graph G we mean any partition of $V(G)$ into independent subsets $C_1, \ldots, C_k$ such that for each $i, j \in \{1, \ldots, k\}$ with $i < j$, each vertex in C_j has a neighbor in C_i. The maximum integer k such that there exists a Grundy coloring with k colors, is called the Grundy number (also called the First-Fit chromatic number) and denoted by $\Gamma(G)$ (also $\chi_{FF}(G)$). It can be observed that $\Gamma(G)$ is equal to the maximum number of colors used by the greedy coloring procedure in G [21]. The literature is full of papers concerning the extremal and algorithmic aspects of the Grundy number e.g. [8, 9, 21]. The Grundy number is an NP-complete quantity even for very restricted families of graphs [21].

By a color-dominating coloring of G we mean any partition of $V(G)$ into independent subsets $C_1, \ldots, C_k$ such that for each i, the class C_i contains a vertex that has a neighbor in every other class C_j, $j \neq i$. Denote by $\mathrm{b}(G)$ the maximum number of colors used in any color-dominating coloring of G and call $\mathrm{b}(G)$ the b-chromatic number of G. It is worth-mentioning that b-chromatic number $\mathrm{b}(G)$ is the worst case behavior of a color suppressing heuristic, based on recoloring all the vertices of a color class, unless there is a vertex in the class having neighbors of every other color. The b-chromatic number has been widely studied in graph theory [6, 10, 12–15]. The b-chromatic number is NP-complete and hard to approximate [5]. A graph G is called b-monotone in [20] if for each induced subgraph H of G we have $\mathrm{b}(H) \leq \mathrm{b}(G)$. This concept is similar to the concept of b-monotonous graphs, defined in [13]. A graph G is b-monotonous if $\mathrm{b}(H_1) \geq \mathrm{b}(H_2)$ for every induced subgraph H_1 of G and every induced subgraph H_2 of H_1. This concept is defined also in [2], in which the graph G is called b-monotonic. Note that every b-monotonous (b-monotonic) graph is also b-monotone. We say a family $\mathcal{F}$ is b-monotone if any graph G of $\mathcal{F}$ is b-monotone.

The comparative study of Grundy and b-chromatic numbers of graphs was initiated by [20]. A natural question concerning comparison of Grundy and b-chromatic numbers is to explore or generate families of graphs $\{G_n\}_{n\geq 1}$ and $\{H_n\}_{n\geq 1}$ such that $\mathrm{b}(G_n) - \Gamma(G_n) \to \infty$ and $\Gamma(H_n) - \mathrm{b}(H_n) \to \infty$. It was proved in [20] that both of the above-mentioned situations happen in the universe of graphs. The following conjecture was made in [20].

Conjecture 1 *There exits a function f such that for any* b*-monotone graph G,* $\Gamma(G) \leq f(\mathrm{b}(G))$.

As proved in [20], the inequality of Conjecture 1 is not valid for the family of non-b-monotone graphs. The papers [17, 18] introduce more families satisfying the conjecture. It was proved in [17] that the conjecture is valid for trees, cacti and some other families in terms of forbidden subgraphs. It was proved in [18] that $\Gamma(G) - \lceil \log \Gamma(G) \rceil \leq \mathrm{b(G)}$, if the girth of G is sufficiently large with respect to its maximum degree. It was also proved in [18] that if G is $K_{2,3}$-free then $\Gamma(G) \leq (\mathrm{b}(G))^3/2$. Given any graph G and a Grundy coloring C for G, in the following we define a subgraph H_C corresponding to G and C. This subgraph will be used in proof of the following results.

Let C be a Grundy coloring for a graph G, define a subgraph H_C corresponding to G and C as follows. Let k be the number of colors in C. Let v_k be a vertex of color k in C. Choose a set $v_1, \ldots, v_{k-1}$ consisting of $k-1$ neighbors of v_k with distinct colors $1, \ldots, k-1$, respectively. Define $L_1 = \{v_k\}$ and $L_2 = \{v_1, \ldots, v_{k-1}\}$. Now, for each i and j with $1 \leq i < j \leq k-1$, any vertex of color j in L_2 needs a neighbor of color i in C. If such a vertex say u is not found in L_2 then put u in a newly defined set L_3. The set L_3 consists only of such vertices. For each $i \in \{2, 3\}$, the presence of a vertex v of color, say j in L_i means that there exists a vertex u in L_{i-1} such that v is adjacent to u and the color of u is greater than j. In this situation v is said to be a child of u and we write $v \in CH(u)$. Also denote the color of any vertex w in C by $c(w)$. Define $H_C = G[L_1 \cup L_2 \cup L_3]$. For a better representation of H_C, we draw H_C in top-down form in which the vertex v_k is the most top vertex and is placed in the first level L_1. The vertices $v_1, \ldots, v_{k-1}$ are in the second level L_2 and so on.

It was proved in [20] that Conjecture 1 is valid for b-monotone $K_{m,n}$-free graphs. But an explicit function f is not obtained in [20]. In this paper we present some comparative results concerning $K_{2,t}$-free graphs by presenting explicit functions f. The extremal properties of $K_{2,t}$-free graphs have been widely studied in graph theory, e.g. [11, 16]. The following proposition shows that for any $t \geq 4$ the family of $K_{2,t}$-free graphs is not b-monotone.

Proposition 1 *(i) Let $t \geq 4$ be arbitrary and fixed integer. There are infinitely many $K_{2,t}$-free graphs which are not* b*-monotone.*
(ii) There does not exist any function f such that $\Gamma(G) \leq f(\mathrm{b}(G))$ for any $K_{2,t}$-free graph G.

Proof To prove (i), define H_t as the graph obtained from $K_{t,t}$ by removing a matching of size $t-1$. Let u and v be the two vertices of degree t in H_t. Note that $\mathrm{b}(H_t) = 2$ and $\mathrm{b}(H_t \backslash \{u, v\}) = t - 1 \geq 3$. Now, let H_t^k be a graph obtained from H_t by attaching k vertices of degree one to u. Let W be the set of vertices of degree one in H_t^k. Note that H_t^k is $K_{2,t}$-free. It can be shown by checking the cases that $\mathrm{b}(H_t^k) = 2$. Also $\mathrm{b}(H_t^k \setminus (W \cup \{u, v\})) = t - 1$. Therefore, H_t^k is not b-monotone.
To prove (ii), note that $\mathrm{b}(H_t^k) = 2$ but $\Gamma(H_t^k) = t + 1$ and t can be arbitrarily large. □

Although Proposition 1 proves existence of infinitely many $K_{2,t}$-free graphs which are not b-monotone, but for any $t \geq 2$, the family of $K_{2,t}$-free graphs includes many infinite b-monotone subfamilies. It was reported in [13] that any graph of girth five is b-monotonous and then b-monotone. Observe that any graph of girth five is also $K_{2,t}$-free, for any $t \geq 2$. A P_4-sparse graph is a graph where every 5-vertex subset contains at most one induced P_4. The so-called P_4-tidy graphs form a superset of P_4-sparse graphs. It was proved in [19] that P_4-tidy graphs are b-monotonic and then b-monotone. It follows that the family of $K_{2,t}$-free P_4-tidy graphs is b-monotone and this family may contain graphs containing $K_{2,q}$ for any $q < t$ since $K_{2,q}$ is P_4-tidy. Using [13] we obtain that $(K_{2,t}, P_5, P, dart)$-free graphs are b-monotone and $K_{2,t}$-free, where P and $dart$ are fixed graphs on five vertices. It was also proved in [3] that graphs with stability (i.e. independence) number at most 2 are b-monotonic. Obviously, any such graph is $K_{2,3}$-free. Hence, our bound for $K_{2,3}$-free graphs applies for graphs of stability number 2.
We have to mention an important comment concerning the conditions imposed on t and $\Gamma(G)$ in the theorems of the paper. First, it was proved in [21] that given any graph G, the condition $k \leq \Gamma(G)$ can be decided by a polynomial-time algorithm in terms of $|V(G)|$, provided that k is a constant integer. In Theorem 1 we assume that $t \geq 2$ is fixed integer and consider b-monotone $K_{2,t}$-free graphs G such that $(2t)^t \leq \Gamma(G)$. Since $(2t)^t$ is a constant value then the validity of condition $(2t)^t \leq \Gamma(G)$ can be checked by a polynomial-time algorithm for any given graph G. If the condition holds then Theorem 1 is applied. Otherwise, i.e. if $\Gamma(G) < (2t)^t$ then obviously $\Gamma(G) < ((2t)^t/2)\mathrm{b}(G)$, i.e. a linear upper bound for $\Gamma(G)$ in terms of $\mathrm{b}(G)$. This comment is also applied for Theorems 2 and 3.
The outline of the paper is as follows. In Sect. 2 we prove in Theorem 1 (i) that if G is any b-monotone $K_{2,t}$-free graph such that $t \leq (\sqrt[t]{\Gamma(G)})/2$ then $\lfloor \sqrt[t]{\Gamma(G)} \rfloor /2 \leq \mathrm{b}(G)$. It is proved in Theorem 1 (ii) that if G is any b-monotone $K_{2,2}$-free graph then $\lfloor (\Gamma(G)+2)^{1/2} \rfloor \leq \mathrm{b}(G)$. In Sect. 3 we prove in Theorem 2 that if G is any b-monotone $(K_{2,t}, K_3)$-free graph, such that $3 \leq t \leq (\Gamma(G)/2) - 2$ then $(\Gamma(G)/2t) + 2 \leq \mathrm{b}(G)$. In Theorem 3 we prove that if G is any b-monotone $(K_{2,t}, K_4)$-free graph, such that $t \leq (5\Gamma(G)/12)^{1/3}$ then $\lfloor (5\Gamma(G)/12)^{1/3} \rfloor \leq \mathrm{b}(G)$.

2 Results for $K_{2,t}$-Free Graphs

This section deals with $K_{2,t}$-free graphs, where $t \geq 2$ is a fixed integer. The following two lemmata shall be used in the proof of Theorem 1.

Lemma 1 *Let p and t be two positive integers with $t \leq p$. Then*

$$\frac{(p+t-2)^{t-1}}{(t-1)!} \leq \frac{2^t p^t - 2p + 3}{p-2}.$$

Proof The inequality holds for $t = 2$ since in this case the inequality simplifies to

$$\frac{(p+t-2)^{t-1}}{(t-1)!} = p \leq \frac{4p^2 - 2p + 3}{p-2}.$$

For the case $t = 3$ the inequality is simply checked.

$$\frac{(p+t-2)^{t-1}}{(t-1)!} = \frac{(p+1)^2}{2} \le \frac{8p^3 - 2p + 3}{p-2} = \frac{2^t p^t - 2p + 3}{p-2}.$$

Let now $t \ge 4$. In the second inequality below we have used $(\frac{n}{e})^n \le n!$ (see [4]).

$$\begin{aligned}\frac{(p+t-2)^{t-1}}{(t-1)!} &\le \frac{(2p-2)^{t-1}}{(t-1)!} = \frac{2^{t-1}(p-1)^{t-1}}{(t-1)!}\\ &\le \frac{2^{t-1}(p-1)^{t-1}}{\frac{(t-1)^{t-1}}{e^{t-1}}} = \frac{2^{t-1}(p-1)^{t-1}e^{t-1}}{(t-1)^{t-1}}\\ &= \left(\frac{p-1}{t-1}e\right)^{t-1} 2^{t-1} < 2^t p^{t-1} - 2 \le 2^t p^{t-1} - 2 + \frac{3}{p}\\ &= \frac{2^t p^t - 2p + 3}{p} \le \frac{2^t p^t - 2p + 3}{p-2}.\end{aligned}$$

We demonstrate the correctness of $\left(\frac{p-1}{t-1}e\right)^{t-1} 2^{t-1} < 2^t p^{t-1} - 2$ as follows.

$$\begin{aligned}\frac{e}{t-1} < 1 &\Longrightarrow \frac{e}{t-1}(p-1) < p\\ &\Longrightarrow \left(\frac{p-1}{t-1}e\right)^{t-1} < p^{t-1}\\ &\Longrightarrow 2^{t-1}\left(\frac{p-1}{t-1}e\right)^{t-1} < 2^{t-1}p^{t-1} < 2^t p^{t-1} - 2.\end{aligned}$$

□

Lemma 2 *Let p and t be two positive integers with $t \le p$. Then*

$$\frac{(p+t-2)^{t-1}}{(t-1)!} \le 2^t p^t - p.$$

Proof

$$\begin{aligned}\frac{(p+t-2)^{t-1}}{(t-1)!} &\le \frac{(2p-2)^{t-1}}{(t-1)!} = \frac{2^{t-1}(p-1)^{t-1}}{(t-1)!}\\ &\le \frac{2^{t-1}(p-1)^{t-1}}{(t-1)^{t-1}/e^{t-1}} = \frac{2^{t-1}(p-1)^{t-1}e^{t-1}}{(t-1)^{t-1}}\end{aligned}$$

$$= \left(\frac{e}{t-1}\right)^{t-1} (p-1)^{t-1}2^{t-1} < (p-1)^{t-1}2^{t-1}$$
$$< (p-1)^{t-1}2^t < p^{t-1}2^t < 2^t p^t - p.$$

□

In the following we use the bound $R(r,s) \le \binom{r+s-2}{s-1} \le \dfrac{(r+s-2)^{s-1}}{(s-1)!}$ (See [1]).

Theorem 1 *Let G be a* b*-monotone $K_{2,t}$-free graph.*
(i) If $3 \le t \le (\sqrt[t]{\Gamma(G)})/2$. Then $\frac{1}{2}\lfloor\sqrt[t]{\Gamma(G)}\rfloor \le \mathrm{b}(G)$.
(ii) If $t = 2$ then $\lfloor(\Gamma(G)+2)^{1/2}\rfloor \le \mathrm{b}(G)$.

Proof We present a same proof for both cases $t \ge 3$ and $t = 2$. Only in some details the proofs are different which are specified in the proof. Let C be a Grundy coloring of G with $k = \Gamma(G)$ colors and H_C be the subgraph corresponding to C. Recall that H_C consists of the levels L_1, L_2, L_3, in which $L_1 = \{v_k\}$ and $L_2 = \{v_1, \ldots, v_{k-1}\}$. For each $i \in \{1, \ldots, k\}$, $c(v_i) = i$ and v_i is adjacent to v_k for each $k \ne i$. In case that $t = 2$, if $2 \le k \le 6$ then $\lfloor\sqrt{\Gamma(G)+2}\rfloor = 2$ and the assertion holds. Assume hereafter that $7 \le \Gamma(G)$ for the case $t = 2$. We define a value p differently for the cases $t \ge 3$ and $t = 2$ as follows.

$$p = \begin{cases} \frac{\lfloor\sqrt[t]{k}\rfloor}{2} & \text{if } t \ge 3, \\ \lfloor\sqrt{k+2}\rfloor & \text{if } t = 2. \end{cases}$$

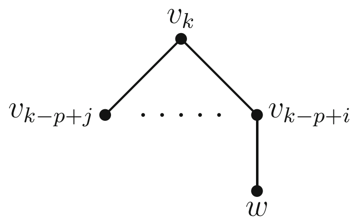
Since G is b-monotone, to prove the theorem it is enough to obtain an induced subgraph of G admitting a b-coloring using p colors. During the proof, we gradually construct a subgraph G' of H_C which is empty at the beginning. Then in each step we "recolor" some vertices of H_C, and add them to G'. If in at least one step of the proof we obtain a complete graph on p vertices then we are done. Otherwise, we prove that the construction of G' is progressed and completed so that the resulting recoloring is a b-coloring with p colors for the induced subgraph G'.

Recoloring Process:

Let G' be an empty set in this step. We introduce a new coloring C', for some vertices of H_C and add them to G'. We use $c'(v)$ to denote the new color of any vertex v. In L_1, recolor v_k by p and add it to G'. In L_2, recolor $v_{k-p+1}, \ldots, v_{k-1}$ by $1, \ldots, p-1$, respectively and add all these vertices to G'. Assume that the vertices of L_3 are presented according to an arbitrary but fixed ordering.

We do the following for any $j \in \{1, \ldots, p-2\}$. If v_{k-1} is adjacent to v_{k-p+j} then the color $k-p+j$ appears in the neighborhood of v_{k-1} and we make nothing. Otherwise, if v_{k-1} is not adjacent to v_{k-p+j}, assign a new color j to the vertex $w \in CH(v_{k-1})$ with $c(w) = k-p+j$ and add w to G'. By doing this, the vertices v_k and v_{k-1} have all colors $1, \ldots, p$ in their closed neighborhood.

Fig. 1 The vertex w with $c(w) = k - p + j$ is added to G' and recolored as $c'(w) = j$



We do the following for any $j \in \{1, \ldots, p-3\}$. If v_{k-2} is adjacent to v_{k-p+j} then the color $k-p+j$ appears in the neighborhood of v_{k-2} and we make nothing. Otherwise, if v_{k-2} is not adjacent to v_{k-p+j}, assign a new color j to the vertex $w \in CH(v_{k-2})$ with $c(w) = k-p+j$ and add w to G'. Until now, v_{k-2} has all colors $1, \ldots, p-2, p$ in its closed neighborhood. If v_{k-2} is also adjacent to v_{k-1}, then color $p-1$ is also in its closed neighborhood in C'. Assume now that v_{k-2} is not adjacent to v_{k-1}.
Claim 1. Either there exists a vertex $x \in CH(v_{k-2})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to v_{k-1} or there is a clique of size p among the vertices of $CH(v_{k-2})$.
Proof of Claim 1. Assume on the contrary that any vertex in $CH(v_{k-2})$ of color in $\{1, \ldots, k-p\}$ is adjacent to v_{k-1}. For $t \geq 3$, using the inequalities $R(p, t) \leq (p+t-2)^{t-1}/(t-1)!$ and $(p+t-2)^{t-1}/(t-1)! \leq k-p$ (by Lemma 2), we obtain that either t independent vertices or a clique of size p among the vertices of $CH(v_{k-2})$ having colors $1, \ldots, k-p$. In the first case, the t independent vertices are adjacent to both vertices v_{k-1} and v_{k-2}. As v_{k-1} and v_{k-2} are not adjacent, we have an induced $K_{2,t}$, a contradiction. Therefore this case does not happen and hence we obtain a clique of size p among the vertices of $CH(v_{k-2})$. This proves Claim 1 for $t \geq 3$.
For $t = 2$, we have the Ramsey bound $R(2, p) = p$, also $p \leq k-p$ holds for any $4 \leq k$. Therefore we obtain either two disjoint vertices or a clique of size p among the vertices of $CH(v_{k-2})$ having colors $1, \ldots, k-p$. In the first case, the two non-adjacent vertices are both adjacent to v_{k-1} and also v_{k-2}. As v_{k-1} and v_{k-2} are not adjacent, we have an induced $K_{2,2}$, a contradiction. Therefore this case does not happen and hence we obtain a clique of size p among the vertices of $CH(v_{k-2})$. This proves Claim 1 for $t = 2$.
In case that there is a clique of size p, this clique is our desired subgraph as its b-chromatic number is p. As the only vertex in G' of color $p-1$ is v_{k-1}, if the other case of Claim 1 holds, we recolor x by $p-1$. By doing this, the vertex v_{k-2} has all the colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper.
In a general case, let i be an arbitrary index with $2 \leq i < p-1$. If v_{k-p+i} is adjacent to v_{k-p+j}, $1 \leq j < i < p-1$, we do nothing. If v_{k-p+i} is not adjacent to v_{k-p+j}, then assign a new color j to the vertex $w \in CH(v_{k-p+i})$ with $c(w) = k-p+j$ and add w to G'. This situation is depicted in Fig. 1.
Until now, all colors $1, 2, \ldots, i-1, i, p$ appear in the closed neighborhood of v_{k-p+i}. If v_{k-p+i} is adjacent to a vertex of G' with color say $m \in \{i+1, \ldots, p-1\}$, then the color m too appears in the closed neighborhood of v_{k-p+i}. Assume hereafter that

there exists some color say m in $\{i+1, \dots, p-1\}$ such that v_{k-p+i} is not adjacent to any vertex of G' of color m.

Claim 2. For each color $m \in \{i+1, \dots, p-1\}$, either there exists a vertex $x \in CH(v_{k-p+i})$ such that $c(x) \in \{1, \dots, k-p\}$ and x is not adjacent to a vertex of G' with color m in C' or there is a clique of size p among the vertices of $CH(v_{k-p+i})$.

Proof of Claim 2. Assume on the contrary that for some $m \in \{i+1, \dots, p-1\}$, any vertex in $CH(v_{k-p+i})$ of primary color in $\{1, \dots, k-p\}$ is adjacent to a vertex of G' whose color is m in C'. We have $p-i-1$ elements in $\{i+1, \dots, p-1\}$ and therefore at most $p-i-2$ vertices of $CH(v_{k-p+i})$ have been recolored with a new color $n \in \{i+1, \dots, p-1\}$, $n \neq m$, and have been added to G' before this step. Therefore among the vertices of $CH(v_{k-p+i})$ of primary color 1 to $k-p$ (which corresponds to a set of $k-p$ vertices), there are still at least $(k-p)-(p-i-2) = k-2p+i+2$ many children, which are not still recolored and added to G'. By the contrary hypothesis, all these $k-2p+i+2$ vertices, are adjacent to a vertex of G' of color m. On the other hand, based on the structure of G', for any vertex $v_j \in G'$, there is at most one child of v_j whose color is m. Therefore there are at most $(k-1)-(k-p+i) = p-i-1$ vertices of color m in G'. Based on the Pigeonhole Principle, at least $\lfloor (k-2p+i+2)/(p-i-1) \rfloor$ many vertices are all adjacent to one vertex of color m.

Now, for $t \geq 3$ we use the following inequalities. The first inequality is valid by Lemma 1 and in the second one we use $i \geq 2$.

$$\frac{(p+t-2)^{t-1}}{(t-1)!} \leq \frac{k-2p+4}{p-3} \leq \frac{k-2p+i+2}{p-i-1} \tag{1}$$

For $t = 2$ we use the following inequalities. In the second one we have used $i \geq 2$.

$$R(2, p) = p \leq \frac{k-2p+4}{p-3} \leq \frac{k-2p+i+2}{p-i-1} \tag{2}$$

In case $t \geq 3$, the inequalities $R(p,t) \leq (p+t-2)^{t-1}/(t-1)!$ and (1) imply that we have either t independent vertices or a clique of size p. This situation is depicted in Fig. 2. The first case does not happen because otherwise it yields an induced $K_{2,t}$ on the t independent vertices, v_{k-p+i} and the vertex of color m. This proves Claim 2 for $t \geq 3$.

In case $t = 2$, the inequalities (2) imply that we have either two non-adjacent vertices or a clique of size p. The first case does not happen since otherwise it yields an induced $K_{2,2}$ on those non-adjacent vertices, v_{k-p+i} and also the vertex of color m. This proves Claim 2 for $t = 2$.

If the second case of Claim 2 holds then we obtain a b-coloring using p colors in the clique. This clique is the desired subgraph. In the case that there exists the vertex x of Claim 2, we recolor x by m and add it to G'. Now, the vertex v_{k-p+i} has all colors $1, \dots, p$ in its closed neighborhood and until this step, the recoloring is proper.

The only vertex remained to be checked is v_{k-p+1}. Note that $c'(v_{k-p+1}) = 1$. If this vertex is adjacent to v_{k-p+j}, $2 \leq j \leq p-1$, we do nothing. Otherwise, assume that v_{k-p+1} is not adjacent to v_{k-p+j}.

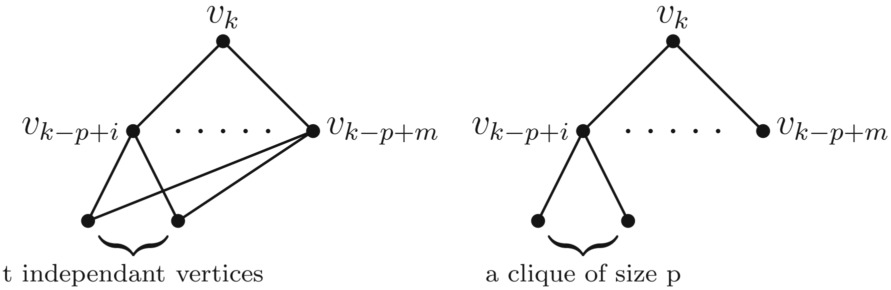


Fig. 2 The two cases discussed in the proof of Claim 2 ($t \geq 3$)

Claim 3. For each color $m \in \{2, \ldots, p-1\}$, we have two possible cases: either there exists a vertex $x \in CH(v_{k-p+1})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to any vertex of G' of color m in C' or there is a clique of size p among the vertices of $CH(v_{k-p+1})$.

Proof of Claim 3. Assume on the contrary that for some $m \in \{2, \ldots, p-1\}$, any vertex in $CH(v_{k-p+1})$ of color in $\{1, \ldots, k-p\}$ is adjacent to a vertex of G' of color m in C'. Since $m \in \{2, \ldots, p-1\}$ then at most $p-3$ vertices among the children of v_{k-p+1} with colors in $\{1, \ldots, k-p\} \setminus \{m\}$, have been added to G' before this step. Therefore there are still $(k-p)-(p-3) = k-2p+3$ children, all adjacent to a vertex in G' of color m. On the other hand, there are at most $(k-1)-(k-p+1) = p-2$ vertices of color m in G'. Based on the Pigeonhole Principle, at least $\lfloor (k-2p+3)/(p-2) \rfloor$ many vertices are all adjacent to one vertex of color m.

For $t \geq 3$, the following inequality holds by Lemma 1.

$$\frac{(p+t-2)^{t-1}}{(t-1)!} \leq \left\lfloor \frac{k-2p+3}{p-2} \right\rfloor.$$

Therefore since $R(p, t) \leq (p+t-2)^{t-1}/(t-1)!$, we have t independent vertices or a clique of size p. As before, the first case does not happen. Hence, Claim 3 is proved for $t \geq 3$.

For $t = 2$, the following inequality holds for any $k > 6$ (i.e. $p > 2$).

$$R(2, p) = p \leq \left\lfloor \frac{k-2p+3}{p-2} \right\rfloor.$$

Therefore we have two independent vertices or a clique of size p. As before, the first case does not happen. This proves Claim 3 for $t = 2$.

In the second case of Claim 3, we obtain a b-coloring using p colors for the clique. This clique is our desired subgraph. In the first case of the claim, we recolor the existing child of v_{k-p+1} by m and add it to G'. The vertex v_{k-p+1} has now all the colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper.

By completing this procedure, either we obtain a clique of size p, at some step, as desired, or in the last step, G' satisfies the properties of our desired subgraph. The new coloring C' is a b-coloring for G' using p colors and the vertices $v_k, v_{k-1}, \ldots, v_{k-p+1}$ form its color-dominating vertices. □

3 $K_{2,t}$-Free Graphs with Some Excluded Cliques

The first theorem concerns $(K_{2,t}, K_3)$-free graphs.

Theorem 2 *Let G be a* b-*monotone $(K_{2,t}, K_3)$-free graph, such that $3 \leq t \leq (\Gamma(G)/2) - 2$. Then*

$$\frac{\Gamma(G)}{2t} + 2 \leq \mathrm{b}(G).$$

Proof Let C be a Grundy coloring of G with $k = \Gamma(G)$ colors and H_C with its levels L_1, L_2, L_3 in which $L_1 = \{v_k\}$ and $L_2 = \{v_1, \ldots, v_{k-1}\}$. For each $i \in \{1, \ldots, k\}$, $c(v_i) = i$ and v_i is adjacent to v_k for each $k \neq i$. Set $p = (k/2t) + 2$. We obtain an induced subgraph G' of H_C which admits a b-coloring with p colors. At the beginning step G' is empty. Then in each step we recolor some vertices of H_C, and add them to G'. The resulting coloring is a b-coloring with p colors for G'.
Recoloring Process:
Let G' be an empty set in this step. We introduce a new coloring C', for some vertices of H_C and add them to G'. We use $c'(v)$ to denote the new color of any vertex v. In L_1, recolor v_k by p and add it to G'. In L_2, recolor $v_{k-p+1}, \ldots, v_{k-1}$ by $1, \ldots, p-1$, respectively and add all these vertices to G'. Assume that the vertices of L_3 are presented according to an arbitrary but fixed ordering.
Vertex v_{k-1} is not adjacent to v_{k-p+j} as there is no K_3 in the graph. We assign a new color j to the vertex $w \in CH(v_{k-1})$ with $c(w) = k-p+j$ and add w to G'. By doing this, the vertices v_k and v_{k-1} have all colors $1, \ldots, p$ in their closed neighborhood.
As before, v_{k-2} is not adjacent to v_{k-p+j}. We assign a new color j to the vertex $w \in CH(v_{k-2})$ with $c(w) = k - p + j$ and add w to G'. Until now, v_{k-2} has all colors $1, \ldots, p-2, p$ in its closed neighborhood.
Claim 1. There exists a vertex $x \in CH(v_{k-2})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to v_{k-1}.
Proof of Claim 1. Assume on the contrary that any vertex in $CH(v_{k-2})$ of color in $\{1, \ldots, k-p\}$ is adjacent to v_{k-1}. we have the following inequalities.

$$R(2, t) = t \leq \frac{k}{2} - 2 = k - \frac{k}{2} - 2 \leq k - \frac{k}{2t} - 2 = k - p.$$

As there is no induced subgraph isomorphic to K_3 in G, there is no edge among the children of v_{k-2}. Therefore we obtain t independent vertices, all adjacent to both vertices v_{k-1} and v_{k-2}. As v_{k-1} and v_{k-2} are not adjacent, we obtain an induced $K_{2,t}$, a contradiction.

As the only vertex in G' of color $p-1$ is v_{k-1}, therefore we recolor x by $p-1$. By doing this, the vertex v_{k-2} has all the colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper. In a general case, v_{k-p+i} is not adjacent to v_{k-p+j}, $1 \leq j < i < p-1$. Assign a new color j to the vertex $w \in CH(v_{k-p+i})$ with $c(w) = k-p+j$ and add w to G'.

Until now, all colors $1, 2, \ldots, i-1, i, p$ appear in the closed neighborhood of v_{k-p+i}. If v_{k-p+i} is adjacent to a vertex of G' with color say $m \in \{i+1, \ldots, p-1\}$, then the color m too appears in the closed neighborhood of v_{k-p+i}. Assume hereafter that there exists some color say m in $\{i+1, \ldots, p-1\}$ such that v_{k-p+i} is not adjacent to any vertex of G' of color m.

Claim 2. For each color $m \in \{i+1, \ldots, p-1\}$, there exists a vertex $x \in CH(v_{k-p+i})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to a vertex of G' with color m in C'.

Proof of Claim 2. Assume on the contrary that for some $m \in \{i+1, \ldots, p-1\}$, any vertex in $CH(v_{k-p+i})$ of primary color in $\{1, \ldots, k-p\}$ is adjacent to a vertex of G' whose color is m in C'. We have $p-i-1$ elements in $\{i+1, \ldots, p-1\}$ and therefore at most $p-i-2$ vertices of $CH(v_{k-p+i})$ have been recolored with a new color $n \in \{i+1, \ldots, p-1\}$, $n \neq m$, and have been added to G' before this step. Therefore among the vertices of $CH(v_{k-p+i})$ of primary color 1 to $k-p$ (which corresponds to a set of $k-p$ vertices), there are still at least $(k-p)-(p-i-2) = k-2p+i+2$ many children, which are not still recolored and added to G'. By the contrary hypothesis, all these $k-2p+i+2$ vertices, are adjacent to a vertex of G' of color m. On the other hand, based on the structure of G', for any vertex $v_j \in G'$, there is at most one child of v_j whose color is m. Therefore there are at most $(k-1)-(k-p+i) = p-i-1$ vertices of color m in G'. Based on the Pigeonhole Principle, at least $\lfloor (k-2p+i+2)/(p-i-1) \rfloor$ many vertices are all adjacent to one vertex of color m. The following inequalities hold. In the second one we have used $i \geq 2$.

$$R(2,t) = t \leq \left\lfloor \frac{k-2p+4}{p-3} \right\rfloor \leq \left\lfloor \frac{k-2p+i+2}{p-i-1} \right\rfloor$$

Consequently, there is t independent vertices among the vertices of $CH(v_{k-p+i})$. This is impossible because otherwise it yields an induced $K_{2,t}$ on the t independent vertices, v_{k-p+i} and the vertex of color m. This contradiction proves Claim 2.

We recolor x by m and add it to G'. Now, the vertex v_{k-p+i} has all colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper. The only vertex remained to be checked is v_{k-p+1}. Note that $c'(v_{k-p+1}) = 1$ and v_{k-p+1} is not adjacent to v_{k-p+j}.

Claim 3. For each color $m \in \{2, \ldots, p-1\}$, there exists a vertex $x \in CH(v_{k-p+1})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to any vertex of G' of color m in C'.

Proof of Claim 3. Assume on the contrary that for some $m \in \{2, \ldots, p-1\}$, any vertex in $CH(v_{k-p+1})$ of color in $\{1, \ldots, k-p\}$ is adjacent to a vertex of G' of color m in C'. Since $m \in \{2, \ldots, p-1\}$ then at most $p-3$ vertices among the children of v_{k-p+1} with colors in $\{1, \ldots, k-p\} \setminus \{m\}$, have been added to G' before this step. Therefore there are still $(k-p)-(p-3) = k-2p+3$ children, all adjacent to a vertex in G' of color m. On the other hand, there are at most $(k-1)-(k-p+1)=p-2$ vertices of color m in G'. Based on the Pigeonhole Principle, at least $\lfloor (k-2p+3)/(p-2) \rfloor$ many vertices are all adjacent to one vertex of color m. The following equalities hold.

$$\left\lfloor \frac{k-2p+3}{p-2} \right\rfloor = \left\lfloor \frac{k-\frac{2k}{2t}-4+3}{\frac{k}{2t}} \right\rfloor = \left\lfloor \frac{2kt-2k-2t}{k} \right\rfloor = \lfloor 2t-2-\tfrac{2t}{k} \rfloor$$
$$= 2t-3$$

For every $t \geq 3$, we have $t \leq 2t-3$ and consequently $R(2,t)=t \leq \left\lfloor \frac{k-2p+3}{p-2} \right\rfloor$. Therefore there are t independent vertices. As before, the case does not happen. Hence, we recolor the existing child of v_{k-p+1} by m and add it to G'. Here the vertex v_{k-p+1} has all the colors $1,\dots,p$ in its closed neighborhood and until this step, the recoloring is proper.

By completing this procedure, we obtain G', satisfying the properties of our desired subgraph. The new coloring C' is a b-coloring for G' using p colors and the set $v_k, v_{k-1}, \dots, v_{k-p+1}$ form its color-dominating vertices. □

Proposition 2 *Let G be a* b*-monotone $(K_{2,2}, K_3)$-free graph. Then*

$$\left\lfloor \frac{\Gamma(G)}{5} \right\rfloor + 2 \leq \mathrm{b}(G)$$

.

Proof If $k=\Gamma(G) \leq 4$, then $\lfloor k/5 \rfloor + 2 = 2$ and obviously $2 \leq \mathrm{b}(G)$ holds. For a $k \geq 5$, we set $p = \lfloor k/5 \rfloor + 2$ and repeat the same procedure as the proof of Theorem 2. The inequalities provided in the claims hold for every $k \geq 5$ as follows.

$$t = 2 \leq \frac{4}{5}k - 2 \leq k-p$$
$$t = 2 = \left\lfloor 3 - \frac{5}{k} \right\rfloor = \left\lfloor \frac{5k-2k-5}{k} \right\rfloor = \left\lfloor \frac{k-\frac{2k}{5}-1}{\frac{k}{5}} \right\rfloor = \left\lfloor \frac{k-2p+3}{p-2} \right\rfloor$$

□

In the following we obtain an inequality concerning $(K_{2,t}, K_4)$-free graphs. In the proof we use the upper bound $R(3,k) \leq (2.4)k^2/\ln k$ proved by Griggs [7].

Theorem 3 *Let G be a* b*-monotone $(K_{2,t}, K_4)$-free graph, such that $t \leq (\frac{5\Gamma(G)}{12})^{1/3}$. Then*

$$\left\lfloor \left(\frac{5\Gamma(G)}{12} \right)^{1/3} \right\rfloor \leq \mathrm{b}(G).$$

Proof The beginning part of the proof is exactly the same as the proof for Theorem 2 except that $p = \lfloor(\Gamma(G)/2.4)^{1/3}\rfloor$. For each $k \leq 19$, we have $p = 1$ and the theorem holds. Assume hereafter that $20 \leq k$. We obtain an induced subgraph G' of H_C which admits a b-coloring with p colors.

Recoloring Process:

At the beginning, define G' as an empty graph. We introduce a new coloring C', for some vertices of H_C and add these vertices to G'. Let $c'(v)$ denote the new color of any vertex v. In L_1, recolor v_k by p and add it to G'. In L_2, recolor $v_{k-p+1}, \ldots, v_{k-1}$ by $1, \ldots, p-1$, respectively and add all these vertices to G'. Assume that the vertices of L_3 are presented according to an arbitrary but fixed ordering.

If v_{k-1} is adjacent to v_{k-p+j}, then v_{k-1} has color j in its neighborhood in G'. If v_{k-1} is not adjacent to v_{k-p+j}, we assign a new color j to the vertex $w \in CH(v_{k-1})$ with $c(w) = k - p + j$ and add w to G'. By doing this, the two vertices v_k and v_{k-1} have all colors $1, \ldots, p$ in their closed neighborhood.

If v_{k-2} is adjacent to v_{k-p+j}, then v_{k-2} has new color j in its neighborhood. In case that v_{k-2} is not adjacent to v_{k-p+j}, assign a new color j to the vertex $w \in CH(v_{k-2})$ with $c(w) = k-p+j$ and add w to G'. Until now, v_{k-2} has all colors $1, \ldots, p-2, p$ in its closed neighborhood. If v_{k-2} is also adjacent to v_{k-1}, then v_{k-2} has color $p-1$ in its closed neighborhood too. Now we assume that v_{k-2} is not adjacent to v_{k-1}.

Claim 1. There exists a vertex $x \in CH(v_{k-2})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to v_{k-1}.

Proof of Claim 1. Assume on the contrary that any vertex in $CH(v_{k-2})$ of color in $\{1, \ldots, k-p\}$ is adjacent to v_{k-1}. As mentioned before the theorem, we have the Ramsey bound $R(3, t) \leq (2.4t^2)/\ln t$. Since $p = \lfloor\sqrt[3]{k/2.4}\rfloor$, we have $2.4p^3 \leq k$. Also $t \leq p$. The following inequalities hold for every $t \geq 3$

$$R(3, t) \leq \frac{2.4t^2}{\ln t} \leq 2.4t^2 \leq 2.4p^2 \leq 2.4p^3 - p \leq k - p$$

Note that in the case $t = 2$, we have $R(3, 2) = 3 \leq (2.4t^2)/\ln t$. If there is a K_3 among the vertices of $CH(v_{k-2})$, then an induced subgraph K_4 is produced in G, a contradiction. Consequently, we obtain t independent vertices all adjacent to both vertices v_{k-1} and v_{k-2}. As v_{k-1} and v_{k-2} are not adjacent, we obtain an induced $K_{2,t}$ in G, a contradiction.

As the only vertex in G' of color $p-1$ is v_{k-1}, we recolor x by $p-1$. By doing this, the vertex v_{k-2} has all the colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper.

In a general case, if v_{k-p+i} is adjacent to v_{k-p+j}, $1 \leq j < i < p-1$, we do nothing. If v_{k-p+i} is not adjacent to v_{k-p+j}, then assign a new color j to the vertex $w \in CH(v_{k-p+i})$ with $c(w) = k - p + j$ and add w to G'.

Until now, all colors $1, 2, \ldots, i-1, i, p$ appear in the closed neighborhood of v_{k-p+i}. If v_{k-p+i} is adjacent to a vertex of G' of color say $m \in \{i+1, \ldots, p-1\}$, then the color m too appears in the closed neighborhood of v_{k-p+i}. Assume hereafter that there exists some color say m in $\{i+1,\ldots,p-1\}$ such that v_{k-p+i} is not adjacent to any vertex of G' of color m.

Claim 2. For each color $m \in \{i+1,\ldots,p-1\}$, there exists a vertex $x \in CH(v_{k-p+i})$ such that $c(x) \in \{1,\ldots,k-p\}$ and x is not adjacent to a vertex of G' with color m in C'.

Proof of Claim 2. Assume on the contrary that for some $m \in \{i+1,\ldots,p-1\}$, any vertex in $CH(v_{k-p+i})$ of primary color in $\{1,\ldots,k-p\}$ is adjacent to a vertex of G' whose color is m in C'. We have $p-i-1$ elements in $\{i+1,\ldots,p-1\}$ and therefore at most $p-i-2$ vertices of $CH(v_{k-p+i})$ have been recolored with a new color $n \in \{i+1,\ldots,p-1\}$, $n \neq m$, and have been added to G' before this step. Therefore among the vertices of $CH(v_{k-p+i})$ of primary color 1 to $k-p$ (which corresponds to a set of $k-p$ vertices), there are still at least $(k-p)-(p-i-2) = k-2p+i+2$ many children, which are not still recolored and added to G'. By the contrary hypothesis, all these $k-2p+i+2$ vertices, are adjacent to a vertex of G' of color m. On the other hand, based on the structure of G', for any vertex $v_j \in G'$, there is at most one child of v_j whose color is m. Therefore there are at most $(k-1)-(k-p+i) = p-i-1$ vertices of color m in G'. Based on the Pigeonhole Principle, at least $\lfloor (k-2p+i+2)/(p-i-1) \rfloor$ many vertices are all adjacent to one vertex of color m. The following inequalities hold. In the second we have used $i \geq 2$.

$$R(3,t) \leq \frac{2.4t^2}{\ln t} \leq \frac{k-2p+4}{p-3} \leq \frac{k-2p+i+2}{p-i-1}$$

If there is a K_3 among the vertices of $CH(v_{k-p+i})$, a K_4 is induced on v_{k-p+i} and 3 of its children. This case does not happen. Therefore a $K_{2,t}$ is induced on the t independent vertices, v_{k-p+i} and the vertex of color m. This case too is a contradiction. This proves Claim 2. We recolor x by m and add it to G'. Now, the vertex v_{k-p+i} has all colors $1,\ldots,p$ in its closed neighborhood and until this step, the recoloring is proper.

The only vertex remained to be checked is v_{k-p+1}. Note that $c'(v_{k-p+1}) = 1$. If v_{k-p+1} is adjacent to v_{k-p+j}, we do nothing. Otherwise, assume that for a j, v_{k-p+1} is not adjacent to v_{k-p+j}.

Claim 3. For each color $m \in \{2,\ldots,p-1\}$, there exists a vertex $x \in CH(v_{k-p+1})$ such that $c(x) \in \{1,\ldots,k-p\}$ and x is not adjacent to any vertex of G' of color m in C'.

Proof of Claim 3. Assume on the contrary that for some $m \in \{2,\ldots,p-1\}$, any vertex in $CH(v_{k-p+1})$ of color in $\{1,\ldots,k-p\}$ is adjacent to a vertex of G' of color m in C'. Since $m \in \{2,\ldots,p-1\}$, at most $p-3$ vertices among the children of v_{k-p+1} with colors in $\{1,\ldots,k-p\}\backslash\{m\}$, have been added to G' before this step. Therefore there are still $(k-p)-(p-3) = k-2p+3$ children, all adjacent to a vertex in G' of color m. On the other hand, there are at most $(k-1)-(k-p+1) = p-2$ vertices of color m in G'. Based on the Pigeonhole Principle, at least $\lfloor (k-2p+3)/(p-2) \rfloor$ many vertices are all adjacent to one vertex of color m. Since $p = \lfloor \sqrt[3]{5k/12} \rfloor$, we have $2.4p^3 \leq k$. Also $t \leq p$. The following inequalities hold for every $t \geq 3$.

$$R(3,t) \leq \frac{2.4t^2}{\ln t} \leq \frac{2.4p^2}{\ln t} \leq 2.4p^2 \leq \frac{k-2p+3}{p-2} - 1 \leq \lfloor \frac{k-2p+3}{p-2} \rfloor$$

The fourth inequality holds as follows

$$
\begin{aligned}
4.8p^2 - 3p + 5 \geq 0 &\Longrightarrow -4.8p^2 \leq -3p + 5 \\
&\Longrightarrow 2.4p^3 - 2(2.4)p^2 \leq 2.4p^3 - 3p + 5 \\
&\Longrightarrow 2.4p^2(p-2) \leq 2.4p^3 - 3p + 5 \\
&\Longrightarrow 2.4p^2 \leq \frac{2.4p^3 - 3p + 5}{p-2} \leq \frac{k - 2p + 3}{p-2} - 1
\end{aligned}
$$

Note that in case $t = 2$, we have $R(3, 2) = 3 \leq (2.4t^2)/\ln(t)$. As before, there is neither K_3 nor t independent vertices among the vertices of $CH(v_{k-p+1})$. Hence, Claim 3 is proved and we recolor the existing child of v_{k-p+1} by m and add it to G'. Now, the vertex v_{k-p+1} has all the colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper.

By completing this procedure, we obtain G', satisfying the properties of our desired subgraph. □

Acknowledgements The authors thank the anonymous reviewers for their useful comments and careful reading of the paper.

Author Contributions All authors contributed to the study conception and design. All authors read and approved the final manuscript.

Funding The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Data Availibility Statement Not applicable.

Declarations

Conflict of Interest The authors declare they have no financial interests and have no relevant financial or non-financial interests to disclose.

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