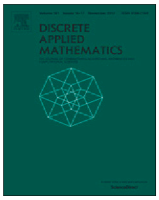
Discrete Applied Mathematics 306 (2022) 1–6

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam



Some comparative results concerning the Grundy and b-chromatic number of graphs

Zoya Masih, Manouchehr Zaker [*]

Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan 45137-66731, Iran

ARTICLE INFO

Article history:
Received 14 February 2021
Received in revised form 23 June 2021
Accepted 14 September 2021
Available online 28 September 2021

Keywords:
Graph coloring
First-Fit coloring
Grundy number
Color-dominating coloring
b-chromatic number
Girth

ABSTRACT

The Grundy and b-chromatic number of graphs are two important chromatic parameters. The Grundy number of a graph G, denoted by $\Gamma(G)$ is the worst case behavior of greedy (First-Fit) coloring procedure for G and the b-chromatic number $\mathrm{b}(G)$ is the maximum number of colors used in any color-dominating coloring of G. Because the nature of these colorings are different they have been studied widely but separately in the literature. In this paper we first prove that $\Gamma(G) - \lceil \log \Gamma(G) \rceil \leq \mathrm{b}(G)$, if the girth of G is sufficiently large with respect to its maximum degree. Next, we prove that if G is $K_{2,3}$-free then $\Gamma(G) \leq (\mathrm{b}(G))^3/2$. These results confirm a previous conjecture for these families of graphs.

1. Introduction

All graphs in this paper are undirected without any loops or multiple edges. We refer to [1] for notations and concepts not defined here. In this paper by $\log k$, we mean $\log_2 k$. By a Grundy coloring of a graph G we mean any partition of $V(G)$ into independent subsets $C_1, \ldots, C_k$ such that for each $i, j \in \{1, \ldots, k\}$ with $i < j$, each vertex in C_j has a neighbor in C_i. The maximum integer k such that there exists a Grundy coloring with k colors, is called the Grundy number (also called the First-Fit chromatic number) and denoted by $\Gamma(G)$ (also $\chi_{FF}(G)$). It can be observed that $\Gamma(G)$ is equal to the maximum number of colors used by the greedy coloring procedure in G [13]. The literature is full of papers concerning the extremal and algorithmic aspects of the Grundy number e.g. [5,6,13,15]. The Grundy number is an NP-complete quantity even for very restricted families of graphs [13]. As proved in [13], for every integer k there exists a unique tree T_k (called tree atom of Grundy number k) such that $\Gamma(T_k) = k$ and T_k is the smallest tree having Grundy number k. Also, for any tree T, $\Gamma(T) \geq k$ if and only if T_k is isomorphic to a subtree of T. In this paper for any graph G and vertex v of G, the closed neighborhood of v, $N[v]$, is the set $N(v)$ of neighbors of v together with v itself.

By a color-dominating coloring of G we mean any partition of $V(G)$ into independent subsets $C_1, \ldots, C_k$ such that for each i, the class C_i contains a vertex that has a neighbor in every other class $C_j, j \neq i$. Denote by $\mathrm{b}(G)$ the maximum number of colors used in any color-dominating coloring of G and call $\mathrm{b}(G)$ the b-chromatic number of G. The b-chromatic number has been widely studied in graph theory [2–4,7–11]. A graph G is called b-monotone in [16] if for each induced subgraph H of G we have $\mathrm{b}(H) \leq \mathrm{b}(G)$. This concept is similar to the concept of quasi-monotonous graphs, introduced in [9]. A graph G is b-monotonous if $\mathrm{b}(H_1) \geq \mathrm{b}(H_2)$ for every induced subgraph H_1 of G and every subgraph H_2 of H_1. Note that every quasi-monotonous graph is also b-monotone. There is a useful quantity denoted by $m(G)$ which is used in study of b-chromatic

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[*] Corresponding author.
E-mail address: mzaker@iasbs.ac.ir (M. Zaker).

https://doi.org/10.1016/j.dam.2021.09.015
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number. Let the vertex degrees of G are ordered in decreasing order $d_1, d_2, \dots, d_n$. Define $m(G) = \max\{k : d_k \geq k-1\}$. It is proved that $\mathrm{b}(G) \leq m(G)$ [8].

The comparative study of Grundy and b-chromatic numbers of graphs was initiated by [16]. A natural question concerning comparison of Grundy and b-chromatic numbers is to explore or generate families of graphs $\{G_n\}_{n\geq 1}$ and $\{H_n\}_{n\geq 1}$ such that $\mathrm{b}(G_n) - \Gamma(G_n) \to \infty$ and $\Gamma(H_n) - \mathrm{b}(H_n) \to \infty$. It was proved in [16] that both of the above-mentioned situations happen in the universe of graphs. The following conjecture was made in [16].

Conjecture 1. *There exists a function f such that $\Gamma(G) \leq f(\mathrm{b}(G))$ for any* b*-monotone graph G.*

As proved in [16], the conjecture is not valid for the family of non b-monotone graphs. The paper [12] introduces more families satisfying the conjecture. By obtaining adequate functions f, it was proved in [12] that the conjecture is valid for trees, cacti and some other families in terms of forbidden subgraphs. Given any graph G and a Grundy coloring C for G, in the following we define a subgraph H_C corresponding to G and C. This subgraph will be used in proof of the following results.

Let G be any graph and C a Grundy coloring of G using say k colors. We define a colored subgraph of G, denoted by H_C as follows. Let v_k be a vertex of color k in C. Choose a set $v_1, \dots, v_{k-1}$ consisting of $k-1$ neighbors of v_k with distinct colors $1, \dots, k-1$, respectively. Define $L_1 = \{v_k\}$ and $L_2 = \{v_1, \dots, v_{k-1}\}$. Now, for each i and j with $1 \leq i < j \leq k-1$, any vertex of color j in L_2 needs a neighbor of color i in C. If such a vertex say u is not found in L_2 then put u in a newly defined set L_3. The set L_3 consists only of such vertices. For a better representation of H_C, we draw H_C in top-down form in which the vertex v_k is the most top vertex and is placed in the first level L_1. The vertices $v_1, \dots, v_{k-1}$ are in the second level L_2 and so on. Repeat the above procedure for the vertices of L_3. Any vertex $v \in L_3$ of color say j needs a neighbor of color i for each $1 \leq i < j$. If such a neighbor w is not found in $L_2 \cup L_3$ then put w in a newly defined set L_4. The set L_4 consists only of such vertices. We continue this procedure and obtain other sets $L_5, \dots$. Let t be an integer such that for any vertex $v \in L_t$, if the color of v in C is say $j > 1$ then for each i with $i < j$, the vertex v has a neighbor of color i in $L_1 \cup \dots \cup L_t$. Now, for each $i \in \{1, \dots, t\}$, define H_i as the subgraph of G induced by $L_1 \cup \dots \cup L_i$. Set also $H_C = G[L_1 \cup \dots \cup L_t]$. We call H_C a subgraph corresponding to the Grundy coloring C. For each $i \geq 1$, the presence of a vertex v of color, say j in L_i means that there exists a vertex u in L_{i-1} such that v is adjacent to u and the color of u is greater than j. In this situation v is said to be a child of u and we write $v \in CH(u)$. Also denote the color of any vertex w in C by $c(w)$.

In this paper we first prove that $\Gamma(G) - \lceil \log \Gamma(G) \rceil \leq \mathrm{b}(G)$, if the girth of G is sufficiently large with respect to its maximum degree. Next, we prove that if G is $K_{2,3}$-free graph then $\Gamma(G) \leq (\mathrm{b}(G))^3/2$. These results confirm Conjecture 1 for these families of graphs.

2. A bound in terms of girth

In [16], a sequence of bipartite graphs $\{B_k\}_{k\geq 1}$ was introduced such that $\mathrm{b}(B_k) = 2$ but $\Gamma(B_k) \to \infty$. This shows that there does not exist any function f such that $\Gamma(G) \leq f(\mathrm{b}(G))$ for all graphs G of girth at least four. It was proved in [12] that if G has girth at least 5 (resp. 6) then $\lfloor \Gamma(G)/2 \rfloor \leq \mathrm{b}(G)$ (resp. $\lfloor 2\Gamma(G)/3 \rfloor \leq \mathrm{b}(G)$). It was also mentioned that the results can be generalized for graphs of girth at least 8 in this form $\lfloor 3\Gamma(G)/4 \rfloor \leq \mathrm{b}(G)$. In this paper we prove that if the girth of graph is sufficiently large with respect to its maximum degree, then a much better upper bound for $\Gamma(G)$ exists in terms of $\mathrm{b}(G)$. Note that as proved in [9], every graph of girth at least five is quasi-monotonous and therefore b-monotone. It was proved in [2] that the b-chromatic number of graphs of girth at least 9 can be obtained by a polynomial time algorithm. Also recall that $\Gamma(G) \leq \Delta(G) + 1$, for every graph G, where $\Delta(G)$ stands for the maximum degree of G.

Theorem 1. *Let G be a graph of girth $g \geq 5$ such that $\Delta(G) + 1 \leq 2^{\frac{g-4}{2}}$. Then $\Gamma(G) - \lceil \log \Gamma(G) \rceil \leq \mathrm{b}(G)$.*

Proof. The condition of the theorem implies that $\log \Gamma(G) \leq \log(\Delta(G) + 1) \leq (g-4)/2$. Let C be a Grundy coloring of G with $k = \Gamma(G)$ colors and H_C be the subgraph of G corresponding to C. Recall that H_C consists of the levels $L_1, \dots, L_s$, for some s, in which $L_1 = \{v_k\}$ and $L_2 = \{v_1, \dots, v_{k-1}\}$. For each $i \in \{1, \dots, k\}$, $c(v_i) = i$ and v_i is adjacent to v_k for each $k \neq i$. Set $t = \lceil \log k \rceil$ and $p = k - t$. Since G is b-monotone, then to prove the theorem it is enough to obtain an induced subgraph of G admitting a b-coloring with p colors. We obtain such an induced subgraph of G by recoloring some portion of G so that the resulting coloring is b-coloring with p colors. After providing the recoloring, we prove that the set of vertices of primary color at least $p-1$ forms a color-dominating set. In the following by "primary color" of a vertex v we mean the color of v in the primary coloring C.

Recoloring Process:

In this step by recoloring some portion of C, we introduce a new coloring C' and we denote the new color of any vertex w in C' by $c'(w)$. Note that, at first, this coloring is not necessarily proper. Later we make some changes to make the coloring proper. In the case $k \in \{1, \dots, 9\}$, we recolor $v_{k-p+1}, \dots, v_{k-1}, v_k$ by $1, \dots, p-1, p$, respectively. In this case the recoloring process is finished in the level L_2. Assume hereafter that $k \geq 10$. Let $S = \{1, 2, \dots, p\}$ be an ordered set of colors to be used in the recoloring process. At each step of the recoloring process, S is updated. In L_1, recolor v_k by p and replace S by $S \setminus \{p\}$. In L_2, recolor $v_{p-1}, \dots, v_{k-1}$ by $2p-k-1, \dots, p-1$, respectively. Replace S by $S \setminus \{2p-k-1, \dots, p-1\}$. The color of other vertices in L_2 remains unchanged. At this stage the recoloring of L_2 is finished.

Now we recolor the vertices of L_3. Assume that the vertices of L_3 are presented according to an arbitrary but fixed ordering. Let w be any vertex of L_3 with color $c(w)$ in the Grundy coloring C. Then w belongs to $CH(v_i)$, for some $v_i \in L_2$. Suppose first that $c(w) \leq p-2$. If $c(w) = c'(v_i)$, then set $c'(w) = p-1$. Otherwise, when $c'(v_i) \neq c(w)$, set $c'(w) = c(w)$. Let now $c(w) \geq p-1$. Note that if $c(w) \geq p-1$ and $c'(v_i) = c(w)$ then $c(w) = p-1$ and we set $c'(w) = c(w)$. In the case that none of the above situations happens, assign the greatest color in S as the new color of w, i.e. $c'(w)$. Update S by removing the color $c'(w)$ from S and go to another vertex in L_3. Note that after all vertices of L_3 are treated, the resulting recoloring has the following property. For each i with $p-1 \leq i \leq k$, the vertex v_i has all colors $1, \ldots, p$ in its closed neighborhood $N[v_i]$.

Now consider the vertices of L_4 according to a fixed ordering. Let y be a vertex in L_4 with color $c(y)$ in the coloring C. Then y belongs to $CH(x)$ for some $x \in L_3$ and x belongs to $CH(v_i)$ for some $v_i \in L_2$. If $c(x) = c'(x)$, do not change the color of y. Suppose that $c(x) \neq c'(x)$. If $c(y) \leq p-2$ and x or v_i are not recolored by $c(y)$, then the color of y remains unchanged. If $c(y) \leq p-2$ and x is recolored by $c(y)$, then define $c'(y) = p$. If $c(y) \leq p-2$ and v_i is recolored by $c(y)$, then we put $c'(y) = p-1$. If none of the above situations happens (i.e. $c(y) \geq p-1$) then assign the greatest color in S to y as its new color. Update S and go to another vertex in L_4. Note that after recoloring of L_4, any recolored vertex u of L_3 with $c(u) \geq p-1$ has all colors $1, \ldots, p$ in its closed neighborhood.

Now we explain a general case L_i, $i \geq 5$, for which the recoloring method is similar to the previous levels. Let y be any vertex in L_i. Then y belongs to $CH(x)$ for some vertex $x \in L_{i-1}$ and x belongs to $CH(w)$ for some $w \in L_{i-2}$. If $c(x) = c'(x)$, the color of y remains unchanged. Suppose that $c(x) \neq c'(x)$. If $c(y) \leq p-2$ and x or w are not recolored by $c(y)$, then its color remains unchanged. If $c(y) \leq p-2$ and x is recolored by $c(y)$, then recolor y by p. If $c(y) \leq p-2$ and w is recolored by $c(y)$ then we recolor y by $p-1$. If none of above situations happens, then assign the greatest color in S to y as its new color. Update S and go to another vertex in L_i. Note that for a vertex $u \in L_{i-1}$ in which $c(u) \geq p-1$ and $c(u) \neq c'(u)$, all colors $1, \ldots, p$ appear in $N[u]$.

We continue the recoloring process until either the set S becomes empty or we reach the level L_{t+2}. Recall that $t = \lceil \log k \rceil$.

Pruning stage:

In this stage we prune (remove) some vertices from the recolored subgraph. Starting from L_3, we remove all vertices w in L_3 satisfying the following properties. For some $i \leq p-2$, $w \in CH(v_i)$ and $w \notin CH(v_j)$, for any $j > p-2$. In general, we remove all vertices $w \in L_i$, $i \geq 4$, satisfying the following properties. For some y with $c(y) \leq p-2$, $w \in CH(y)$ but $w \notin CH(y')$, for any y' with $p-2 < c(y')$. The pruning process is continued until at most the level L_{t+2}. Hence, for each $i \in \{3, \ldots, t+2\}$, the level L_i is pruned. Denote by L'_i the remaining vertices in L_i.

Properness of the coloring:

We first claim that the only situation where two vertices are adjacent in the first $t+2$ levels of H_C is if one is a child of another. To prove this statement, suppose on the contrary that there are two distinct adjacent vertices $u, v \in L_1 \cup \cdots \cup L_{t+2}$ such that no one is a child of another. Therefore, according to the structure of H_C, we will have a cycle $\mathcal{C} = u, \ldots, v_k, \ldots, v, u$ in H_C. The length of this cycle is at most $2(t+1)+1$ and hence $g \leq 2t+3$. The latter inequality together with the hypothesis of theorem i.e. $t \leq (g-4)/2$, yield a contradiction. This contradiction proves our claim.

Therefore, the recoloring remains proper unless after recoloring a vertex w has a same color of its child v. According to the rules of the recoloring, such a situation may happen only when the vertex v with $c(v) \geq p-1$ does not receive a new color from S but its parent w receives a new color equal to $c(v)$. In this situation we remove v from L'_i. Otherwise, at the end of the recoloring process no vertex receives a color equal to the color of its child. Consequently, the coloring C' is proper until the level L'_{t+2}. Denote by D the set of all vertices $w \in L_1 \cup L_2 \cup \cdots \cup L_{t+2}$ with $c(w) \geq p-1$ and $c(w) \neq c'(w) \in S$. Denote by G' the subgraph of G induced on the remaining vertices in $L'_1 \cup L'_2 \cup \cdots \cup L'_t$. Note that for each vertex $w \in D$, any color from $\{1, 2, \ldots, p\}$ appears in $N[w]$ of G'. To complete the proof, it suffices to show $|D| \geq p$. By proving this fact, we obtain that G' admits a b-coloring using p colors, as desired.

Claim: $|D| \geq p$.

Proof of claim: If we visit p vertices of primary color at least $p-1$ before the level L_{t+2}, we clearly have $|D| \geq p$. Otherwise, we count the vertices in $L'_1 \cup L'_2 \cup \cdots \cup L'_{t+2}$ of primary color $\geq p-1$. To do so, we first show by induction on $i \geq 2$ that for any color j, $p-1 \leq j \leq k-1$, there are $\binom{k-j-1}{i-2}$ many vertices of primary color j in the level $L'_i \cap D$. We first prove the validity of the induction hypothesis for $i = 2$.

Regarding to the recoloring of vertices in L'_2, note that $v_{p-1}, \ldots, v_{k-1}$ satisfy membership in D. In other words, there are exactly one vertex of every color $p-1 \leq j \leq k-1$ in $L'_2 \cap D$. This proves the induction assertion for $i = 2$.

Assume now that the induction hypothesis holds in all levels $L'_1, \ldots, L'_i$. We prove it for the level $i+1$. We have $\binom{k-j-1}{i-2}$ vertices of color j in $L'_i \cap D$. We have to show that there are $\binom{k-j-1}{i-1}$ vertices of color j in $L'_{i+1} \cap D$.

Note that all colors from $p-1$ to $k-i+1$ appear in $L'_i \cap D$. On the other hand, each vertex in L'_i having a color from $\{j+1, \ldots, k-i+1\}$ introduces one vertex of color j to be put in $L'_{i+1} \cap D$. It follows by the induction hypothesis that the number of vertices of color j in $L'_{i+1} \cap D$ is

$$\underbrace{\binom{k-j-2}{i-2}}_{\#\text{ vertices of color } j+1} + \underbrace{\binom{k-j-3}{i-2}}_{\#\text{ vertices of color } j+2} + \cdots + \underbrace{\binom{i-2}{i-2}}_{\#\text{ vertices of color } k-i+1} = \binom{k-j-1}{i-1}.$$

For any i, there are $\binom{k-j-1}{i-2}$ many vertices in $L'_i \cap D$ of primary color $p-1 \leq j$. Note that all colors $p-1, \ldots, k-i+1$ exist in $L'_i \cap D$. This argument shows that there are overall

$$\sum_{j=p-1}^{k-i+1} \binom{k-j-1}{i-2} = \binom{k-p+1}{i-1}$$

many vertices in $L'_i \cap D$. We end this process in L'_{t+2}. Note that no vertex which we removed in the pruning process, belongs to D. Hence removal of these vertices does not decrease the cardinality of D. We sum the number of vertices in levels $L'_1, \ldots, L'_{t+2}$ and obtain

$$|D| = \sum_{i=1}^{t+2} \binom{k-p+1}{i-1}.$$

Putting $k-p=t$ we obtain

$$|D| = \sum_{i=1}^{t+2} \binom{t+1}{i-1} = 2^{t+1}.$$

Recall that $t = \lceil \log k \rceil$. We have now $p = k-t < 2^{t+1} = |D|$, as desired. □

We have two comments concerning Theorem 1. First, the bound of Theorem 1 is sharp for trees for which the girth is infinity. Let T_k be the tree atom of Grundy number k. It was proved in [12] that $\mathrm{b}(T_k) \leq k - \lfloor \log k \rfloor + 1$. The second comment is that the condition $\Delta(G) + 1 \leq 2^{\frac{g-4}{2}}$ in Theorem 1 can be weakened as follows. For any graph G, define $\Delta_2(G) = \max_{u \in V(G)} \max_{v \in N(u): d(v) \leq d(u)} d(v)$, where $d(v)$ is the degree of v in G. It was proved in [14] that $\Gamma(G) \leq \Delta_2(G) + 1$. Note that the latter bound is much better than the $\Gamma(G) \leq \Delta(G) + 1$ bound for many classes of graphs such as star graphs. We conclude that if G is any graph of girth $g \geq 5$ satisfying $\Delta_2(G) + 1 \leq 2^{\frac{g-4}{2}}$ then $\Gamma(G) - \lceil \log \Gamma(G) \rceil \leq \mathrm{b}(G)$.

3. $K_{2,3}$-free graphs

By a $K_{m,n}$-free graph, we mean any graph containing no induced subgraph isomorphic to the complete bipartite graph $K_{m,n}$. It was proved in [16] that for any arbitrary but fixed positive integers m and n, Conjecture 1 is valid for b-monotone $K_{m,n}$-free graphs. But an explicit function f is not provided in [16]. In the following theorem we obtain an explicit function for $K_{2,3}$-free graphs. By the Ramsey number $R(3,p)$ we mean the minimum integer n such that any graph on at least n vertices contains either a complete subgraph on p vertices or three independent vertices.

Theorem 2. *Let G be a* b*-monotone, $K_{2,3}$-free graph. Then*

$$\lfloor \sqrt[3]{2\Gamma(G)} \rfloor \leq \mathrm{b}(G).$$

Proof. Let C be a Grundy coloring of G with $k = \Gamma(G)$ colors and H_C be the subgraph corresponding to C. Recall that H_C consists of the levels $L_1, \ldots, L_s$, for some s, in which $L_1 = \{v_k\}$ and $L_2 = \{v_1, \ldots, v_{k-1}\}$. For each $i \in \{1, \ldots, k\}$, $c(v_i) = i$ and v_i is adjacent to v_k for each $k \neq i$. Set $p = \lfloor \sqrt[3]{2\Gamma(G)} \rfloor$. Since G is b-monotone, to prove the theorem it is enough to obtain an induced subgraph of G admitting a b-coloring using p colors. In fact, we introduce a subgraph G' of H_C which is empty at the beginning. Then in each step we recolor some vertices of H_C, and add them to G'. The resulting coloring is a b-coloring with p colors.

Recoloring Process:

Let G' be an empty set in this step. We introduce a new coloring C' for some vertices of H_C and add these vertices to G'. We use $c'(v)$ to denote the new color of any vertex v. In L_1, recolor v_k by p and add it to G'. In L_2, recolor $v_{k-p+1}, \ldots, v_{k-1}$ by $1, \ldots, p-1$, respectively and add all these vertices to G'. Assume that the vertices of L_3 are presented according to an arbitrary but fixed ordering.

If v_{k-1} is adjacent to v_{k-p+j}, $1 \leq j \leq p-2$, we do nothing. If v_{k-1} is not adjacent to v_{k-p+j}, then assign a new color j to the vertex $w \in CH(v_{k-1})$ with $c(w) = k-p+j$ and add w to G'. By doing this, the vertices v_k and v_{k-1} have all colors $1, \ldots, p$ in their closed neighborhood.

If v_{k-2} is adjacent to v_{k-p+j}, $1 \leq j \leq p-3$, we do nothing. If v_{k-2} is not adjacent to v_{k-p+j}, then assign a new color j to the vertex $w \in CH(v_{k-2})$ with $c(w) = k-p+j$ and add w to G'. Until now, v_{k-2} has all colors $1, \ldots, p-2, p$ in its closed neighborhood. If v_{k-2} is adjacent to v_{k-1}, then obviously v_{k-2} has color $p-1$ in its closed neighborhood in C'. Otherwise, consider the case that v_{k-2} is not adjacent to v_{k-1}.

Claim 1. Either there exists a vertex $x \in CH(v_{k-2})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to v_{k-1} or there is a clique of size p among the vertices of $CH(v_{k-2})$.

Proof of Claim 1. Assume on the contrary that any vertex in $CH(v_{k-2})$ of color in $\{1, \ldots, k-p\}$ is adjacent to v_{k-1}. Using the upper bound $R(3,p) \leq p(p+1)/2$ and also $p(p+1)/2 \leq k-p$, we obtain either three independent vertices or a clique

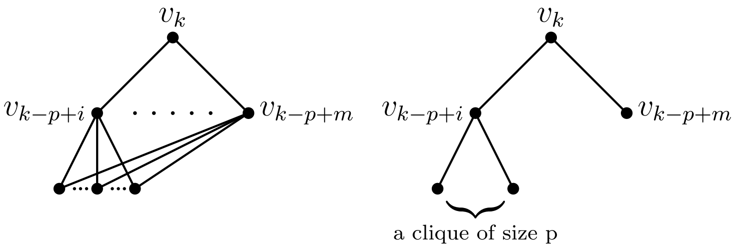

Fig. 1. The two cases discussed in the proof of Claim 2.

of size p among the vertices of $CH(v_{k-2})$ having color $1, \ldots, k-p$. In the first case, three independent vertices are adjacent to both vertices v_{k-1} and v_{k-2}. As v_{k-1} and v_{k-2} are not adjacent, we have an induced $K_{2,3}$, a contradiction. Therefore this case does not happen and hence we obtain a clique of size p among the vertices of $CH(v_{k-2})$. This proves Claim 1.

In case that there is a clique of size p, this clique is our desired subgraph as its b-chromatic number is p. As the only vertex in G' of color $p-1$ is v_{k-1}, therefore if the other case of Claim 1 holds, we recolor x by $p-1$. By doing this, the vertex v_{k-2} has all the colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper.

In a general case, if v_{k-p+i} is adjacent to v_{k-p+j}, $1 \leq j < i < p-1$, we do nothing. If v_{k-p+i} is not adjacent to v_{k-p+j}, then assign a new color j to the vertex $w \in CH(v_{k-p+i})$ with $c(w) = k-p+j$ and add w to G'.

Until now, all colors $1, 2, \ldots, i-1, i, p$ appear in the closed neighborhood of v_{k-p+i}. If v_{k-p+i} is adjacent to a vertex of G' with color say $m \in \{i+1, \ldots, p-1\}$, then the color m too appears in the closed neighborhood of v_{k-p+i}. Assume hereafter that v_{k-p+i} is not adjacent to at least one vertex of G' with color in $\{i+1, \ldots, p-1\}$.

Calim 2. For each color $m \in \{i+1, \ldots, p-1\}$, either there exists a vertex $x \in CH(v_{k-p+i})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to any vertex of G' of color m in C' or there is a clique of size p among the vertices of $CH(v_{k-p+i})$.

Proof of Claim 2. Assume on the contrary that for some $m \in \{i+1, \ldots, p-1\}$, any vertex in $CH(v_{k-p+i})$ of color in $\{1, \ldots, k-p\}$ is adjacent to a vertex of G' whose color is m in C'. There are at most $p-i-2$ vertices of $CH(v_{k-p+i})$ with color $n \in \{i+1, \ldots, p-1\}$, $n \neq m$, which have been added to G' before this step.

Therefore there are still at least $(k-p)-(p-i-2) = k-2p+i+2$ many children, all adjacent to a vertex of G' with color m. On the other hand, based on the structure of G', for any vertex $v_j \in G'$, there is at most one child of v_j whose color is m. Therefore there are at most $(k-1)-(k-p+i) = p-i-1$ vertices of color m in G'. Based on the pigeonhole principle, at least $\lfloor (k-2p+i+2)/(p-i-1) \rfloor$ many vertices are all adjacent to one vertex of color m. The following inequalities hold. In the second we have used $i \geq 2$.

$$\frac{p(p+1)}{2} \leq \frac{k-2p+4}{p-3} \leq \frac{k-2p+i+2}{p-i-1}$$

Hence, using the upper bound $R(3,p) \leq p(p+1)/2$, we have either three independent vertices or a clique of size p (see Fig. 1). The first case does not happen because otherwise it yields an induced $K_{2,3}$ on the three independent vertices, v_{k-p+i} and the vertex of color m. This proves Claim 2.

In the case that there exists the vertex x of Claim 2, we recolor x by m and add it to G'. Now, the vertex v_{k-p+i} has all colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper. If the second case of Claim 2 holds then we obtain a b-coloring using p colors in the clique. This clique is the desired subgraph.

The only vertex remained to be checked is v_{k-p+1}. Note that $c'(v_{k-p+1}) = 1$. If this vertex is adjacent to v_{k-p+j}, $2 \leq j \leq p-1$, we do nothing. Suppose that v_{k-p+1} is not adjacent to v_{k-p+j}.

Claim 3. For each color $m \in \{2, \ldots, p-1\}$, either there exists a vertex $x \in CH(v_{k-p+1})$ such that $c(x) \in \{1, \ldots, k-p\}$ and x is not adjacent to any vertex of G' of color m in C' or there is a clique of size p among the vertices of $CH(v_{k-p+1})$.

Proof of Claim 3. Assume on the contrary that for some $m \in \{2, \ldots, p-1\}$, any vertex in $CH(v_{k-p+1})$ of color in $\{1, \ldots, k-p\}$ is adjacent to a vertex of G' of color m in C'. Since $m \in \{2, \ldots, p-1\}$ then at most $p-3$ vertices among the children of v_{k-p+1} with colors in $\{1, \ldots, k-p\} \setminus \{m\}$, have been added to G' before this step.

Therefore there are still $(k-p)-(p-3) = k-2p+3$ children, all adjacent to a vertex in G' of color m. On the other hand, there are at most $(k-1)-(k-p+1) = p-2$ vertices of color m in G'. Based on the pigeonhole principle, at least $\lfloor (k-2p+3)/(p-2) \rfloor$ many vertices are all adjacent to one vertex of color m. The following inequality holds.

$$\frac{p(p+1)}{2} \leq \left\lfloor \frac{k-2p+3}{p-2} \right\rfloor.$$

Therefore as $R(3,p) \leq p(p+1)/2$, we have three independent vertices or a clique of size p. As before, the first case does not happen. Hence, we recolor the existing child of v_{k-p+1} by m and add it to G'. Here the vertex v_{k-p+1} has all the colors $1, \ldots, p$ in its closed neighborhood and until this step, the recoloring is proper. In the second case, we can obtain a b-coloring with p colors to the clique. This clique is our desired subgraph.

By completing this procedure, either we obtain a clique of size p, at some step, as desired, or in the last step, G' satisfies the properties of our desired subgraph. The new coloring C' is a b-coloring for G' with p colors in which the vertices $v_k, v_{k-1}, \ldots, v_{k-p+1}$ are color-dominating vertices. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors thank the anonymous reviewers of the paper for their useful comments.

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